INTRODUCTION TO TURBULENCE MODELLING
FOCUS: TURBULENCE-VISCOSITY MODELS

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MASTER THESIS

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Introduction

So, **what** is turbulence?

We are surrounded by this phenomena in our day by day live but, do we really understand its behaviour?

Examples:

- Rivers and Waterfalls
- Cigarettes
- Atmospheres
- Oceanic currents
- ...

After more than 100 years of research on the topic we can give a positive answer to the previous question. But in fact, there is still not an analytical and universal way for modelling turbulence due to its chaotic behaviour.

[ESA, Copernicus EU, Sentinel-2]

Fundamentals of Turbulence

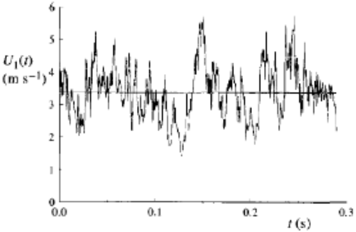
An **essential** feature:

- Fluid field varies significantly and irregularly in both position and time. The velocity field is denoted by $U(x, t)$ where x is the position and t is time.

Turbulent Jet Example:

It is observved that $U_1(t)$ and its mean $\langle U_1 \rangle$ are in some senses 'stable': huge variations in $U_1(t)$are not observed; neither does $U_1(t)$ depend long periods of time near values different than $\langle U_1 \rangle$.

One has to know that in a turbulent flow the random velocity field $U(x, t)$ is a much more mathematical object than the single random variable U. Statistical treatment of the phenomenon is a must.



[Stephen B. Pope, Turbulent Flows, Cambridge]

Equations of Fluid Motion

For our purposes, only N-S continuity and momentum equations are explained.

Continuity Equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho U) = 0 \tag{1}$$

Represents the mass conservation principle. For an inviscid flow $\nabla \cdot U = 0$.

Momentum Equation

$$\frac{d}{dt}(\rho \overrightarrow{v}) + \nabla \cdot (\rho \overrightarrow{v}\,\overrightarrow{v}) = -\nabla p + \nabla \cdot (\overrightarrow{\tau}) + \rho \overrightarrow{g} \tag{2}$$

With S. B. Pope approach:

$$\frac{DU}{Dt} = -\frac{1}{\rho}\nabla P + \nu \nabla^2 U \tag{3}$$

Hypothesis

- Continuity of matter
- Continuum medium assumption
- Relativity effects negligible
- Inertial reference system
- Magnetic and electromagnetic forces negligible

Special treatment required for the non-linear term of the N-S equations:

$$\nabla \cdot (\rho \overrightarrow{v}\,\overrightarrow{v}) \tag{4}$$

Mass Conservation Equation

$$\boxed{\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho U) = 0} \qquad (5)$$

Equation Terms	Description
Continuity	This equation defines that the variation of mass in the control volume has to be equal to the mass flow through its faces.
$\frac{d\rho}{dt}$	Variation of mass inside the control volume in a differential time.
$\nabla \cdot (\rho \vec{V})$	Mass flow through the faces of the control volume.

Momentum Conservation Equation

$$\frac{d}{dt}(\rho \overrightarrow{V}) + \nabla \cdot (\rho \overrightarrow{V}\,\overrightarrow{V}) = -\nabla p + \nabla \cdot (\overrightarrow{\tau}) + \rho \overrightarrow{g} \qquad (6)$$

Momentum	This equation shows us that the variation of linear momentum in the control volume plus the momentum flux through the CV faces has to be equal to the sum of the forces that act on the CV.
$\frac{d}{dt}(\rho \overrightarrow{V})$	Represents the variation of linear momentum in the control volume.
$\nabla \cdot (\rho \overrightarrow{V}\,\overrightarrow{V})$	Represents the momentum flux through the faces of its control volume.
∇p	Pressure gradient acting like an axial force on the faces of the cv.
$\nabla \cdot (\overrightarrow{\tau})$	Total stress tensor. This force acts axially and tangentially on the faces of the control volume. Its value depends on the type of fluid (Newtonian, non-Newtonian...).
$\rho \overrightarrow{g}$	Volumetric force. This force may be a gravitational, electrical, magnetic or electromagnetic.

Energy Conservation Equation

$$\frac{d}{dt}(\rho(u+e_c))+\nabla\cdot((u+e_c)\rho\overrightarrow{V}) = -\nabla\cdot(\rho\overrightarrow{V})+\nabla(\overrightarrow{V}\cdot\overrightarrow{\tau})-\nabla\cdot\overrightarrow{q}+\rho\overrightarrow{g}\cdot\overrightarrow{V}+G \quad (7)$$

Energy	Defines that the variation of internal energy and kinetic energy in a control volume plus the flow of their variables must be equal to the work done on the control volume plus the incoming heat flow through the faces of the CV plus the energy of the sources in the control volume.
$\frac{d}{dt}(\rho(u+e_c))$	Represents the variation of the internal and kinetic energy in the cv.
$\nabla\cdot((u+e_c)\rho\overrightarrow{V})$	Represents the energy flow of these variables through the faces of its volume.
$-\nabla\cdot(\rho\overrightarrow{V})$	Work done by superficial forces like pressure and stress.
$\nabla\cdot\overrightarrow{q}$	Work done by superficial forces like pressure and stress.
$\nabla(\overrightarrow{V}\cdot\overrightarrow{\tau})$	Incoming heat flow through the faces of the control volume.
$\nabla\cdot\overrightarrow{q}$	Represents the work done by the volumetric forces, in this case there is only the gravitational force work.
$\rho\overrightarrow{g}\cdot\overrightarrow{V}$	Work done by the internal forces.

Simplified N-S Equations

Hypotheses:
- 2D model
- Laminar flow
- Incompressible flow
- Newtonian fluid
- Boussinesq hypothesis
- Negligible viscous dissipation
- Negligible compression or expansion work
- Non-participating medium in radiation
- Mono-component and mono-phase fluid

Continuity Equation

$$\frac{du}{dx} + \frac{dv}{dy} = 0 \tag{8}$$

Momentum equation

$$\rho\frac{du}{dx} + \rho u\frac{du}{dx} + \rho v\frac{du}{dy} = -\frac{dp}{dx} + \mu\left(\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2}\right) \tag{9}$$

$$\rho\frac{du}{dx}+\rho u\frac{dv}{dx}+\rho v\frac{dv}{dy} = -\frac{dp}{dy}+\mu\left(\frac{d^2v}{dx^2}+\frac{d^2v}{dy^2}\right)+\rho g\beta(T-T_\infty) \tag{10}$$

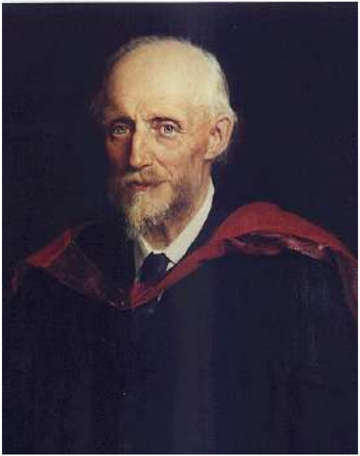
Reynolds Equations

Recalling the fluctuating velocity concept. The decomposition of the velocity $U(x,t)$ into its mean $\langle U(x,t)\rangle$ and the fluctuation

$$u(x,t) \equiv U(x,t) - \langle U(x,t)\rangle \tag{11}$$

is referred to as the **Reynolds decomposition**, i.e.,

$$\boxed{U(x,t) = \langle U(x,t)\rangle + u(x,t)} \tag{12}$$



Osborne Reynolds FRS (23 August 1842 – 21 February 1912) was an innovator in the understanding of fluid dynamics. He spent his entire career at what is now the University of Manchester.

Continuity Equation

It follows from the continuity equation (Eq. 5)

$$\nabla \cdot U = \nabla \cdot (\langle U \rangle + u) \tag{13}$$

which gives

$$\boxed{\nabla \cdot \langle U \rangle = 0} \tag{14}$$

and by substraction we obtain

$$\boxed{\nabla \cdot u = 0} \tag{15}$$

Momentum Equation

Taking the mean of the momentum equation (Eq. 3) is less simple because of the nonlinear convective term. The first step is to write the substantial derivative in conservative form

$$\frac{DU_j}{Dt} = \frac{\partial U_j}{\partial t} + \frac{\partial}{\partial x_i}(U_i U_j) \tag{16}$$

so that the mean is

$$\left\langle \frac{DU_j}{Dt} \right\rangle = \frac{\partial \langle U_j \rangle}{\partial t} + \frac{\partial}{\partial x_i} \langle U_i U_j \rangle \tag{17}$$

Then, substituting the Reynolds decomposition for U_i and U_j, the non linear term becomes $\langle U_i U_j \rangle = \langle U_i \rangle \langle U_j \rangle + \langle u_i u_j \rangle$. The velocity covariances $\langle u_i u_j \rangle$ are called **Reynolds stresses**. Thus, from the previous to equations it is obtained

$$\boxed{\left\langle \frac{DU_j}{Dt} \right\rangle = \frac{\partial \langle U_j \rangle}{\partial t} + \frac{\partial \langle U_i \rangle \langle U_j \rangle}{\partial x_i} + \frac{\partial}{\partial x_i} \langle u_i u_j \rangle} \tag{18}$$

For any property $Q(x,t)$, $\overline{D}Q/\overline{D}t$ represents its rate of change following a point moving with the local mean velocity $\langle U(x,t)\rangle$. In terms of this derivative, Eq. 18 is

$$\left\langle \frac{DU_j}{Dt} \right\rangle = \frac{\overline{D}}{\overline{D}t}\langle U_j\rangle + \frac{\partial}{\partial x_i}\langle u_i u_j\rangle \tag{19}$$

Introducing the mean of Eq. (3) into the previous equation, the result is the mean-momentum or **Reynolds equations**

$$\frac{\overline{D}\langle U_j\rangle}{\overline{D}t} = \nu\nabla^2\langle U_j\rangle - \frac{\partial\langle u_i u_j\rangle}{\partial x_i} - \frac{1}{\rho}\frac{\partial\langle p\rangle}{\partial x_j} \tag{20}$$

Introducing Eq. (18) and Eq. (24) into Eq. (19) leads to the most common form of the equation:

$$\boxed{\frac{\partial\langle U_j\rangle}{\partial t} + \frac{\partial\langle U_i\rangle\langle U_j\rangle}{\partial x_i} = \nu\nabla^2\langle U_j\rangle - \frac{\partial\langle u_i u_j\rangle}{\partial x_i} - \frac{1}{\rho}\frac{\partial\langle p\rangle}{\partial x_j}} \tag{21}$$

Rearranging the Eq. (21) reynolds stress and viscosity terms leads to the expression:

$$\boxed{\frac{\partial\langle U_j\rangle}{\partial t}+\frac{\partial\langle U_i\rangle\langle U_j\rangle}{\partial x_i}=\nu\nabla^2\langle U_j\rangle-\frac{\partial\langle u_iu_j\rangle}{\partial x_i}-\frac{1}{\rho}\frac{\partial\langle p\rangle}{\partial x_j}} \tag{22}$$

Which reminds us to the formulation of the incompressible momentum equation with the stress tensor τ_{ij} as

$$\frac{\partial\langle U_j\rangle}{\partial t}+\frac{\partial\langle U_i\rangle\langle U_j\rangle}{\partial x_i}=\boxed{\frac{\partial}{\partial x_i}\left(\nu\frac{\partial\langle U_j\rangle}{\partial x_i}-\tau_{ij}\right)}-\frac{1}{\rho}\frac{\partial\langle p\rangle}{\partial x_j} \tag{23}$$

Reynolds Stresses

The Reynolds equation Eq. (24) can be rewriten

$$\rho \frac{\overline{D}\langle U_j\rangle}{\overline{D}t} = \frac{\partial}{\partial x_i}\left[\mu\left(\frac{\partial\langle\langle U_i\rangle\rangle}{\partial x_j} + \frac{\partial\langle\langle U_j\rangle\rangle}{\partial x_i}\right) - \rho\langle u_i u_j\rangle - \langle p\rangle\delta_{ij}\right] \qquad (24)$$

There are also other important definitions:
- **Turbulent kinetic energy** $k(x,t)$:

$$k \equiv \frac{1}{2}\langle u \cdot u\rangle = \frac{1}{2}\langle u_i u_i\rangle \qquad (25)$$

- **Anisotropy**

$$a_{ij} \equiv \langle u_i u_j\rangle - \frac{2}{3}k\delta_{ij} \qquad (26)$$

Anisotropy tensor

$$b_{ij} = \frac{a_{ij}}{2k} \qquad (27)$$

In terms of anisotropy tensors the Reynolds stress

$$\langle u_i u_j\rangle = \frac{2}{3}k\delta_{ij} + a_{ij} = 2k(\frac{1}{3}\delta_{ij} + b_{ij}) \qquad (28)$$

We can identify three different stresses from the terms in the square brackets:

- **Viscous stress**:

$$\mu\left(\frac{\partial\langle\langle U_i\rangle\rangle}{\partial x_j} + \frac{\partial\langle\langle U_j\rangle\rangle}{\partial x_i}\right)$$

- **Isotropic stress**:

$$-\langle p\rangle\delta_{ij}$$

- **Aparent stresses**:

$$-\rho\langle u_i u_j\rangle$$

Statistics and Averaging

We know for the velocity $U(x,t)$ that it is decomposed into its mean and the fluctuation as $U(x,t) = \langle U(x,t)\rangle + u(x,t)$. From the previous presentation, there are two different assembling methods for the statistical parameters:

- **Time averaging**:

$$\boxed{\langle U(x,t)\rangle = lim_{T\to\infty}\frac{1}{T}\int_0^T U(x,t)dt} \tag{29}$$

Used for the RANS simulations.

- **Ensemble averaging**:

$$\boxed{\langle U(x,t)\rangle = \frac{1}{Dt}\int_t^{t+Dt} U(x,t)dt} \tag{30}$$

For the URANS calculations the averaging occurs over a time interval Dt.

Within the ensemble averaging the stochastic process is repeated N times from the initial state.The ensemble averaging is then:

$$\langle U(x,t^*)\rangle = lim_{N\to\infty}\frac{1}{N}\sum_{i=1}^{N} U(x,t) \tag{31}$$

Modelling and Simulation

As it is known, there are three main modelling methodologies:

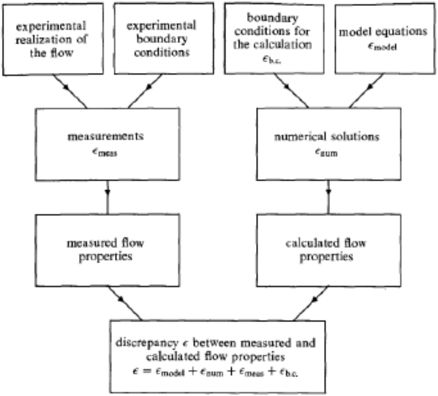
- **DNS**
- **LES**
- **RANS**

Since the thesis is focused on URANS compressible modelling, the RANS method is deeply analyzed.

From S.B. Pope, we have a list of the principal criteria that can be used to asses different models

(i) level of description,
(ii) completeness,
(iii) cost and ease of use,
(iv) range and applicability, and
(v) accuracy

The following chart helps to define the accuracy of a model:



[Stephen B. Pope, Turbulent Flows, Cambridge]

Turbulence-Viscosity Models

The Reynolds stresses which appear as unknown in the Reynolds equations are determined by by a turbulence model via the **turbulent viscosity hypothesis**. According to the hyp. the reynolds stresses are given by:

$$\boxed{\langle u_i u_j \rangle = \frac{2}{3} k \delta_{ij} - \nu_T \left(\frac{\partial \langle U_i \rangle}{\partial x_j} + \frac{\partial \langle U_j \rangle}{\partial x_i} \right)} \tag{32}$$

Where ν_T is the turbulent viscosity field. Relating the anisotropy from Eq. (26)

$$a_{ij} = \langle u_i u_j \rangle - \frac{2}{3} k \delta_{ij} = -\nu_T \left(\frac{\partial \langle U_i \rangle}{\partial x_j} + \frac{\partial \langle U_j \rangle}{\partial x_i} \right) \tag{33}$$

k is the turbulence kinetic energy. Including the strain-rate tensor $\overline{S}_{ij}$:

$$\overline{S}_{ij} = \frac{1}{2} \left(\frac{\partial \langle U_i \rangle}{\partial x_j} + \frac{\partial \langle U_j \rangle}{\partial x_i} \right) \quad \rightarrow \quad \boxed{a_{ij} = -2\nu_T \overline{S}_{ij}} \tag{34}$$

From the previous relationships, the Reynolds stresses can be defined as:

$$\langle u_i u_j \rangle = \frac{2}{3} k \delta_{ij} - 2\nu_T \overline{S}_{ij} \tag{35}$$

Recalling the Eq. (22) but rearranging some terms,

$$\frac{\partial \langle U_j \rangle}{\partial t} + \frac{\partial \langle U_i \rangle \langle U_j \rangle}{\partial x_i} = \boxed{\frac{\partial}{\partial x_i}\left(\nu \frac{\partial \langle U_j \rangle}{\partial x_i} - \langle u_i u_j \rangle \right)} - \frac{1}{\rho}\frac{\partial \langle p \rangle}{\partial x_j} \tag{36}$$

It is easy to see now that if we include the turbulent viscosity hypothesis into the momentum equation then a model for the viscosity would be required.

The models are commonly defined by its number of equations, having one-equation and two-equation models such as the $k - \varepsilon$ one.

The $k - \varepsilon$ Model

The $k - \varepsilon$ model belongs to the class of **two-equation** models. It is the most widely used model and was developed by Lauder and Sharma (1974). It mainly consist of:

(i) the model transport equation of k which consists on the one-equation model transport equation:

$$\frac{\overline{D}k}{\overline{D}t} = \nabla \cdot \left(\frac{\nu_T}{\sigma_k} \nabla k \right) + \wp - \varepsilon \tag{37}$$

Where the Prandtl number σ_k is usually considered as $\sigma_k = 1$.

(ii) the model transport equation for ε which is mainly empirical

$$\frac{\overline{D}\varepsilon}{\overline{D}t} = \nabla \cdot \left(\frac{\nu_T}{\sigma_\varepsilon} \nabla \varepsilon \right) + C_{\varepsilon 1} \frac{\wp \varepsilon}{k} - C_{\varepsilon 2} \frac{\varepsilon^2}{k} \tag{38}$$

The standard values from the Launder and Sharma (1974) are

$$C_\mu = 0.09, \quad C_{\varepsilon 1} = 1.44, \quad C_{\varepsilon 2} = 1.92, \quad \sigma_k = 1.0, \quad \sigma_\varepsilon = 1.3 \tag{39}$$

(iii) the specification of the turbulent viscosity as

$$\boxed{v_T = \frac{C_\mu k^2}{\varepsilon}} \tag{40}$$

Where $C_\mu = 0.09$ is one of five model constants.

Recalling that:

$$\frac{\overline{D}}{\overline{D}t} \equiv \frac{\partial}{\partial t} + \langle U \rangle \cdot \nabla \tag{41}$$

we can rewrite the k and ε Eq. 42 - 45 model transport equations as:

$$\boxed{\frac{\partial k}{\partial t} + \langle U \rangle \cdot \frac{\partial k}{\partial x_i} = \nabla \cdot \left(\frac{\nu_T}{\sigma_k} \nabla k \right) + \wp - \varepsilon} \tag{42}$$

$$\boxed{\frac{\partial \varepsilon}{\partial t} + \langle U \rangle \cdot \frac{\partial \varepsilon}{\partial x_i} = \nabla \cdot \left(\frac{\nu_T}{\sigma_\varepsilon} \nabla \varepsilon \right) + C_{\varepsilon 1} \frac{\wp \varepsilon}{k} - C_{\varepsilon 2} \frac{\varepsilon^2}{k}} \tag{43}$$

where $\wp$ is the rate of production of the turbulent kinetic energy:

$$\wp = -\langle u_i u_j \rangle \frac{\partial \langle U_j \rangle}{\partial x_i} \tag{44}$$

The $k - \omega$ Model

The $k - \omega$ model is an extension from the previous one. For homogeneous turbulence the choice of the second variable is immaterial but for inhomogeneous flows, the difference lies in the **diffusion term**.

The $k - \omega$ model models the second term using a combination from the previous model such as $\omega \equiv \varepsilon / k$. The model equation is:

$$\boxed{\frac{\partial \omega}{\partial t} + \langle U \rangle \cdot \frac{\partial \omega}{\partial x_i} = \nabla \cdot \left(\frac{\nu_T}{\sigma_\omega} \nabla \omega \right) + C_{\omega 1} \frac{\wp \omega}{k} - C_{\omega 2} \frac{\omega^2}{k}} \tag{45}$$

The Shear Stress Transport (SST) Model

The idea behind the SST model is to retain the robust and accurate formulation of the Wilcox $k-\omega$ model in the near wall region, and to take advantage of the freestream independence of the $k-\varepsilon$ model in the outer part of the boundary layer. [2]

To achieve this, the $k-\varepsilon$ model is transformed into a $k-\omega$ formulation. The original model is then multiplied by a function F_1 and the transformed model by a function $(1-F_1)$, and both are added together. The function F_1 will be designed to be one in the near wall region (activating the original model) and zero away from the surface.[2]

Original k-ω model:

$$\frac{D\rho k}{Dt} = \tau_{ij}\frac{\partial u_i}{\partial x_j} - \beta^* \rho\omega k + \frac{\partial}{\partial x_j}\left[(\mu + \sigma_{k1}\mu_t)\frac{\partial k}{\partial x_j}\right] \qquad (1)$$

$$\frac{D\rho\omega}{Dt} = \frac{\gamma_1}{\nu_t}\tau_{ij}\frac{\partial u_i}{\partial x_j} - \beta_1\rho\omega^2 + \frac{\partial}{\partial x_j}\left[(\mu + \sigma_{\omega 1}\mu_t)\frac{\partial \omega}{\partial x_j}\right] \qquad (2)$$

Transformed k-ϵ model:

$$\frac{D\rho k}{Dt} = \tau_{ij}\frac{\partial u_i}{\partial x_j} - \beta^* \rho\omega k + \frac{\partial}{\partial x_j}\left[(\mu + \sigma_{k2}\mu_t)\frac{\partial k}{\partial x_j}\right] \qquad (3)$$

$$\frac{D\rho\omega}{Dt} = \frac{\gamma_2}{\nu_t}\tau_{ij}\frac{\partial u_i}{\partial x_j} - \beta_2\rho\omega^2 + \frac{\partial}{\partial x_j}\left[(\mu + \sigma_{\omega 2}\mu_t)\frac{\partial \omega}{\partial x_j}\right] + 2\rho\sigma_{\omega 2}\frac{1}{\omega}\frac{\partial k}{\partial x_j}\frac{\partial \omega}{\partial x_j} \qquad (4)$$

Now, Eq. (1) and Eq. (2) are multiplied by F_1 and Eq. (3) and Eq. (4) are multiplied by $(1-F_1)$ and the corresponding equations of each set are added together to give the new model:

$$\frac{D\rho k}{Dt} = \tau_{ij}\frac{\partial u_i}{\partial x_j} - \beta^* \rho\omega k + \frac{\partial}{\partial x_j}\left[(\mu + \sigma_{k}\mu_t)\frac{\partial k}{\partial x_j}\right] \qquad (5)$$

$$\frac{D\rho\omega}{Dt} = \frac{\gamma}{\nu_t}\tau_{ij}\frac{\partial u_i}{\partial x_j} - \beta\rho\omega^2 + \frac{\partial}{\partial x_j}\left[(\mu + \sigma_{\omega}\mu_t)\frac{\partial \omega}{\partial x_j}\right] + 2\rho(1-F_1)\sigma_{\omega 2}\frac{1}{\omega}\frac{\partial k}{\partial x_j}\frac{\partial \omega}{\partial x_j} \qquad (6)$$

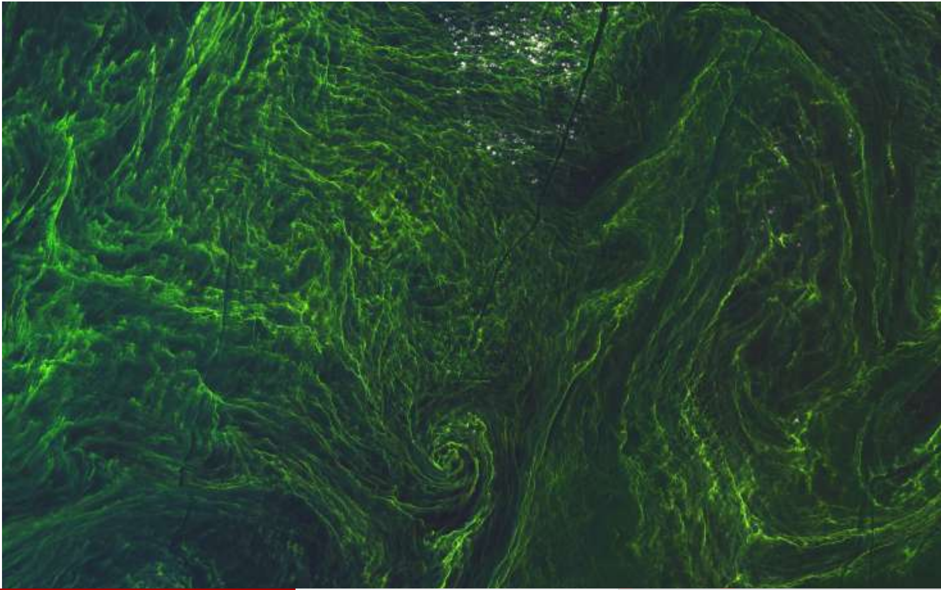
Conclusions

- It should be clear now how the Turbulence-Viscosity models work. The momentum equation has been fully understood and it is clear how the model defines the turbulent viscosity.

- The $k - \varepsilon$ model in its standard form fails badly in capturing the main features of flows with vortex shedding. In particular, the model severely underestimates the magnitude of the fluctuations in the pressure field resulting in the underprediction of the root-mean-square values of the lift and drag coefficients. [1]

- Consideration of the SST model could really improve the potential of the solution.

Questions?



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[2] F. R. Menter, Two-Equation Eddy-Viscosity Turbulence Models for Engineering Applications

[3] This Sentinel-2 image from August 2020 shows algal bloom in the Baltic Sea.